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New ergodic convergence theorems for non-expansive mappings and *m*-accretive mappings

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Abstract

Two new ergodic convergence theorems for approximating the common element of the set of zero points of an *m*-accretive mapping and the set of fixed points of an infinite family of non-expansive mappings in a real smooth and uniformly convex Banach space are obtained, which improves some of the previous work. The computational experiments to demonstrate the effectiveness of the proposed iterative algorithms in this paper are conducted.

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1 Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use '$\rightarrow$' and '$\rightharpoonup$' (or 'w-lim') to denote strong and weak convergence either in E or in E^*, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

A Banach space E is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x-y\| \geq \varepsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

A Banach space E is said to be smooth if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\}$.

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx := \left\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\}, \quad x \in E.$$

If E is reduced to the Hilbert space H, then $J \equiv I$ is the identity mapping. If E is smooth, then $J : E \to E^*$ is norm-to-norm continuous. The normalized duality mapping J is said to be weakly sequentially continuous at zero if $\{x_n\}$ is a sequence in E which converges weakly to 0; it follows that $\{Jx_n\}$ converges in weak* to 0.

For a mapping $T : D(T) \sqsubseteq E \to E$, we use $\mathrm{Fix}(T)$ to denote the fixed point set of it; that is, $\mathrm{Fix}(T) := \{x \in D(T) : Tx = x\}$.

Let $T : D(T) \sqsubseteq E \to E$ be a mapping. Then T is said to be:

(1) non-expansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for } \forall x, y \in D(T);$$

(2) contraction if there exists $0 < k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\| \quad \text{for } \forall x, y \in D(T);$$

(3) accretive if, for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\bigl\langle Tx - Ty, j(x - y)\bigr\rangle \geq 0;$$

(4) m-accretive if T is accretive and $R(I + \lambda T) = E$ for $\forall \lambda > 0$;

(5) strongly positive (see [1]) if $D(T) = E$ where E is a real smooth Banach space and there exists $\overline{\gamma} > 0$ such that

$$\langle Tx, Jx\rangle \geq \overline{\gamma}\|x\|^2 \quad \text{for } \forall x \in E;$$

in this case,

$$\|aI - bT\| = \sup_{\|x\| \leq 1} \bigl|\bigl\langle (aI - bT)x, J(x)\bigr\rangle\bigr|,$$

where I is the identity mapping and $a \in [0, 1]$, $b \in [-1, 1]$;

(6) demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx = p$.

For the accretive mapping A, we use $A^{-1}0$ to denote the set of zero points of it; that is, $A^{-1}0 := \{x \in D(A) : Ax = 0\}$. If A is accretive, then we can define, for each $r > 0$, a non-expansive single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$, which is called the resolvent of A [2]. It is well known that J_r^A is non-expansive and $A^{-1}0 = \mathrm{Fix}(J_r^A)$.

Let C be a nonempty, closed, and convex subset of E and Q be a mapping of E onto C. Then Q is said to be sunny [3] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$.

A mapping Q of E into E is said to be a retraction [3] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q.

A subset C of E is said to be a sunny non-expansive retract of E [3] if there exists a sunny non-expansive retraction of E onto C and it is called a non-expansive retract of E if there exists a non-expansive retraction of E onto C.

The first mean ergodic theorem for non-expansive mappings was proved by Baillon in Hilbert space [4]. That is, for each $x \in C$, the Cesaro means

$$S_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} T^k x, \quad n \geq 1, \tag{1.1}$$

converge weakly to a fixed point of a non-expansive mapping T.

The implicit midpoint rule (IMR) for non-expansive mappings in a Hilbert space was introduced by [5]. This rule generates a sequence $\{x_n\}$ via the semi-implicit procedure:

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_{n+1} + x_n}{2}\right), \quad n \geq 0. \tag{1.2}$$

$\{x_n\}$ is proved to be weakly convergent to a fixed point of the non-expansive mapping T.

The ergodic convergence of the sequence $\{x_n\}$ generated by (1.2) is considered in a Hilbert space in [6]. That is, the convergence of the means

$$z_n := \frac{1}{\sum_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k x_k, \quad n = 1, 2, \ldots, \tag{1.3}$$

where $\{x_k\}$ satisfies (1.2), to a fixed point of the non-expansive mapping T is obtained.

In a Hilbert space, Marino *et al.* presented the following iterative algorithm in [7]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.4}$$

where f is a contraction, A is a strongly positive linear bounded operator, and T is non-expansive. If $\operatorname{Fix}(T) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to $p \in \operatorname{Fix}(T)$, which solves the variational inequality $\langle (\gamma f - A)p, z - p \rangle \leq 0$, for $\forall z \in \operatorname{Fix}(T)$ under some conditions.

Motivated by the previous work, we shall present two iterative algorithms and the ergodic convergence theorems are obtained. The connection between the strongly convergent point and the solution of one kind variational inequalities is being set up. Some examples are exemplified to illustrate the effectiveness of the proposed algorithms. The computational experiments are conducted and the codes are written in Visual Basic Six.

We need the following preliminaries.

Lemma 1.1 (see [8]) *Let E be a Banach space and C be a nonempty closed and convex subset of E. Let $f : C \to C$ be a contraction. Then f has a unique fixed point $u \in C$.*

Lemma 1.2 (see [9]) *Let E be a real uniformly convex Banach space, C be a nonempty, closed, and convex subset of E and $T : C \to E$ be a non-expansive mapping such that* $\operatorname{Fix}(T) \neq \emptyset$, *then $I - T$ is demiclosed at zero.*

Lemma 1.3 (see [10]) *In a real Banach space E, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 1.4 (see [11]) *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\} \subset (0,1)$ *and* $\{b_n\}$ *is a number sequence. Assume that* $\sum_{n=0}^{\infty} t_n = +\infty$, $\limsup_{n\to\infty} \frac{b_n}{t_n} \le 0$, *and* $\sum_{n=0}^{\infty} c_n < +\infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5 (see [12]) *Let E be a Banach space and let A be an m-accretive mapping. For* $\lambda > 0$, $\mu > 0$, *and* $x \in E$, *one has*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x \right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ *and* $J_\mu^A = (I + \mu A)^{-1}$.

Lemma 1.6 (see [13]) *Assume T is a strongly positive bounded operator with coefficient* $\overline{\gamma} > 0$ *on a real smooth Banach space E and* $0 < \rho \le \|T\|^{-1}$. *Then* $\|I - \rho T\| \le 1 - \rho\overline{\gamma}$.

Lemma 1.7 (see [14]) *Let E be a real smooth and uniformly convex Banach space and C be a nonempty, closed, and convex sunny non-expansive retract of E, and let Q_C be the sunny non-expansive retraction of E onto C. Let $f : E \to E$ be a fixed contractive mapping with coefficient* $k \in (0,1)$, $T : E \to E$ *be a strongly positive linear bounded operator with coefficient* $\overline{\gamma}$ *and* $U : C \to C$ *be a non-expansive mapping. Suppose that the duality mapping* $J : E \to E^*$ *is weakly sequentially continuous at zero,* $0 < \eta < \frac{\overline{\gamma}}{2k}$ *and* $\mathrm{Fix}(U) \neq \emptyset$. *If for each* $t \in (0,1)$, *define* $T_t : E \to E$ *by*

$$T_t x := t\eta f(x) + (I - tT)UQ_C x,$$

then T_t has a fixed point x_t, for each $0 < t \le \|T\|^{-1}$, *which is convergent strongly to the fixed point of U, as $t \to 0$. That is,* $\lim_{t\to 0} x_t = p_0 \in \mathrm{Fix}(U)$. *Moreover, p_0 satisfies the following variational inequality: for* $\forall z \in \mathrm{Fix}(U)$,

$$\bigl\langle (T - \eta f)p_0, J(p_0 - z)\bigr\rangle \le 0.$$

Lemma 1.8 (see [15]) *Let E be a real strictly convex Banach space and let C be a nonempty closed and convex subset of E. Let* $T_m : C \to C$ *be a non-expansive mapping for each* $m \ge 1$. *Let* $\{a_m\}$ *be a real number sequence in* $(0,1)$ *such that* $\sum_{m=1}^{\infty} a_m = 1$. *Suppose that* $\bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m) \neq \emptyset$. *Then the mapping* $\sum_{m=1}^{\infty} a_m T_m$ *is non-expansive with* $\mathrm{Fix}(\sum_{m=1}^{\infty} a_m T_m) = \bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m)$.

2 Strong convergence theorems

Lemma 2.1 *Suppose* $A : D(A) \subset E \to E$ *is an m-accretive mapping, where E is a Banach space, and* $\{r_n\}_{n=0}^{\infty} \subset (0,+\infty)$ *is any real number sequence. Then, for* $n \ge 0$, $\forall x, y \in E$, *if* $r_n \le r_{n+1}$,

$$\bigl\|J_{r_{n+1}}^A x - J_{r_n}^A y\bigr\| \le \|x - y\| + \frac{r_{n+1} - r_n}{r_{n+1}} \bigl\|J_{r_{n+1}}^A x - y\bigr\|; \tag{2.1}$$

if $r_{n+1} \le r_n$,

$$\bigl\|J_{r_{n+1}}^A x - J_{r_n}^A y\bigr\| \le \|x - y\| + \frac{r_n - r_{n+1}}{r_n} \bigl\|x - J_{r_n}^A y\bigr\|. \tag{2.2}$$

Proof In fact, if $r_n \leq r_{n+1}$, then, using Lemma 1.5, we have

$$\begin{aligned}\|J_{r_{n+1}}^A x - J_{r_n}^A y\| &= \left\| J_{r_n}^A \left[\frac{r_n}{r_{n+1}} x + \left(1 - \frac{r_n}{r_{n+1}}\right) J_{r_{n+1}}^A x \right] - J_{r_n}^A y \right\| \\ &\leq \frac{r_n}{r_{n+1}} \|x - y\| + \left(1 - \frac{r_n}{r_{n+1}}\right) \|J_{r_{n+1}}^A x - y\| \\ &\leq \|x - y\| + \frac{r_{n+1} - r_n}{r_{n+1}} \|J_{r_{n+1}}^A x - y\|,\end{aligned}$$

which implies that (2.1) is true.

If $r_{n+1} \leq r_n$, then imitating the proof of (2.1), we have (2.2).

This completes the proof. □

2.1 Ergodic convergence of the first iterative algorithm

Theorem 2.2 *Let E be a real smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny non-expansive retract of E, and Q_C be the sunny non-expansive retraction of E onto C. Let $f : C \to C$ be a contraction with contractive constant $k \in (0,1)$, $T : C \to C$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Let $A : C \to E$ be an m-accretive mapping, and $S_i : C \to C$ be non-expansive mappings, for $i = 1, 2, \ldots$. Let $\Omega := \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap A^{-1}0 \neq \emptyset$. Suppose the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero, and $0 < \eta < \frac{\overline{\gamma}}{2k}$. Suppose $\{\alpha_{n,i}\}$, $\{\delta_n\}$, $\{\beta_n\}$, $\{b_{n,i}\}$, $\{\gamma_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$, where $n \geq 0$ and $i = 1, 2, \ldots$. Let $\{x_n\}$ be generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) \sum_{i=1}^{\infty} b_{n,i}[(1 - \alpha_{n,i}) J_{r_n}^A + \alpha_{n,i} S_i] Q_C x_n, \\ u_n = (1 - \delta_n) y_n + \delta_n J_{r_n}^A (\frac{u_n + y_n}{2}), \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T) u_n, \quad n \geq 0, \end{cases} \tag{A}$$

where $J_{r_n}^A = (I + r_n A)^{-1}$, for $n \geq 0$. Further suppose that the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \sum_{i=1}^{\infty} |\alpha_{n+1,i} - \alpha_{n,i}| < +\infty$;

(ii) $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\gamma_n \to 0$, *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < +\infty$;

(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < +\infty$ *and* $0 < \varepsilon \leq r_n$, *for* $n \geq 0$;

(iv) $\sum_{n=1}^{\infty} |\delta_n - \delta_{n-1}| < +\infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < +\infty$, $\beta_n \to 0$, $\delta_n \to 0$, *as* $n \to \infty$;

(v) $\sum_{n=0}^{\infty} \sum_{i=1}^{\infty} |b_{n+1,i} - b_{n,i}| < +\infty$ *and* $\sum_{i=1}^{\infty} b_{n,i} = 1$ *for* $n \geq 0$.

Then the three sequences $\{y_n\}$, $\{u_n\}$, and $\{x_n\}$ converge strongly to the unique element $q_0 \in \Omega$ which satisfies the following variational inequality: *for* $\forall y \in \Omega$,

$$\langle (T - \eta f) q_0, J(q_0 - y) \rangle \leq 0. \tag{2.3}$$

Moreover, the ergodic convergence is obtained in the sense that

$$z_n := \frac{1}{\sum_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k x_k, \quad n \geq 1,$$

converges strongly to the above q_0, under the assumption that $\{a_n\}$ is a sequence of positive numbers such that $\sum_{k=1}^{n} a_k \to \infty$, as $n \to \infty$.

To prove Theorem 2.2, we need the following lemmas.

Lemma 2.3 *In Theorem* 2.2, *set* $W_{n,i} = (1-\alpha_{n,i})J_{r_n}^A + \alpha_{n,i}S_i$. *Then* $W_{n,i} : C \to C$ *is non-expansive for* $n \geq 0$ *and* $i = 1,2,\ldots$. *Moreover,* $\Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_{n,i})$.

Proof It can be easily obtained that $W_{n,i} : C \to C$ is non-expansive since both $J_{r_n}^A$ and S_i are non-expansive, for $n \geq 0$ and $i = 1,2,\ldots$.

The fact that $\Omega \subset \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_{n,i})$ is trivial. We are left to show that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(W_{n,i}) \subset \Omega$.

For $p \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_{n,i})$, then $p = (1-\alpha_{n,i})J_{r_n}^A p + \alpha_{n,i}S_i p$. For $\forall q \in \Omega \subset \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_{n,i})$, we have

$$\begin{aligned} \|p-q\| &= \big\|(1-\alpha_{n,i})\big(J_{r_n}^A p - J_{r_n}^A q\big) + \alpha_{n,i}(S_i p - S_i q)\big\| \\ &\leq (1-\alpha_{n,i})\big\|J_{r_n}^A p - q\big\| + \alpha_{n,i}\|S_i p - q\| \\ &\leq (1-\alpha_{n,i})\big\|J_{r_n}^A p - q\big\| + \alpha_{n,i}\|p-q\|, \end{aligned}$$

which implies that $\|J_{r_n}^A p - q\| = \|p-q\|$. Similarly, $\|p-q\| = \|S_i p - q\|$. Since E is uniformly convex, we have $p - q = J_{r_n}^A p - q = S_i p - q$, which implies that $p \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap A^{-1}0$.

This completes the proof. □

Lemma 2.4 *In Theorem* 2.2, $\{z_n\}$ *is well-defined.*

Proof In fact, it suffices to show that $\{u_n\}$ is well-defined.

For $t \in (0,1)$, define $U_t : C \to C$ by $U_t x := (1-t)y + tU(\frac{x+y}{2})$, where $U : C \to C$ is non-expansive for $x, y \in C$. Then for $\forall x, y \in C$,

$$\|U_t x - U_t z\| \leq t\left\|\frac{x+y}{2} - \frac{y+z}{2}\right\| \leq \frac{t}{2}\|x-z\|.$$

Thus U_t is a contraction, which ensures from Lemma 1.1 that there exists $x_t \in C$ such that $U_t x_t = x_t$. That is, $x_t = (1-t)y + tU(\frac{y+x_t}{2})$.

Note that $J_{r_n}^A$ is non-expansive, then $\{u_n\}$ is well-defined, and then $\{x_n\}$ and $\{z_n\}$ are all well-defined.

This completes the proof. □

Lemma 2.5 *The variational inequality* (2.3) *in Theorem* 2.2 *has a unique solution in* Ω.

Proof Using Lemmas 1.7, 1.8 and 2.3, we know that there exists v_t such that $v_t = t\eta f(v_t) + (I - tT)\sum_{i=1}^{\infty} b_{n,i}W_{n,i}Q_C v_t$, for $t \in (0,1)$, where $W_{n,i}$ is the same as that in Lemma 2.3, for $i = 1,2,\ldots$. Moreover, $v_t \to q_0 \in \Omega$, as $t \to 0$, which is the unique solution of the variational inequality (2.3).

This completes the proof. □

Proof of Theorem 2.2 Step 1. $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ are all bounded.

For $\forall p \in \Omega$, noticing Lemma 2.3, we see that for $n \geq 0$,

$$\|y_n - p\| \leq \|x_n - p\|. \tag{2.4}$$

Also,

$$\|u_n - p\| \leq (1-\delta_n)\|y_n - p\| + \delta_n \left\| \frac{y_n + u_n}{2} - p \right\| \leq \left(1 - \frac{\delta_n}{2}\right)\|y_n - p\| + \frac{\delta_n}{2}\|u_n - p\|,$$

which implies that

$$\|u_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|. \tag{2.5}$$

Using Lemma 1.6 and (2.5), we have, for $n \geq 0$,

$$\begin{aligned} \|x_{n+1} - p\| &\leq \gamma_n \eta k \|x_n - p\| + \gamma_n \|\eta f(p) - Tp\| + (1 - \gamma_n \overline{\gamma})\|u_n - p\| \\ &\leq \left[1 - \gamma_n(\overline{\gamma} - k\eta)\right]\|x_n - p\| + \gamma_n \|\eta f(p) - Tp\|. \end{aligned} \tag{2.6}$$

By using the inductive method, we can easily get the following result from (2.6):

$$\|x_{n+1} - p\| \leq \max\left\{ \|x_0 - p\|, \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta} \right\}.$$

Therefore, $\{x_n\}$ is bounded. Then both $\{y_n\}$ and $\{u_n\}$ are bounded in view of (2.5).

Moreover, we can easily know that $\{J_{r_n}^A Q_C x_n\}$, $\{S_i Q_C x_n\}$, $\{J_{r_n}^A(\frac{u_n+y_n}{2})\}$, $\{Q_C x_n\}$, $\{f(x_n)\}$, and $\{W_{n,i} Q_C x_n\}$ are all bounded, for $n \geq 0$ and $i = 1, 2, \ldots$.

Set

$$\begin{aligned} M' = \sup\Bigg\{ &\|u_n\|, \|x_n\|, \|y_n\|, \|S_i Q_C x_n\|, \left\| J_{r_n}^A\left(\frac{u_n + y_n}{2}\right)\right\|, \left\|J_{r_n}^A Q_C x_n\right\|, \|Q_C x_n\|, \left\|f(x_n)\right\|, \\ &\|W_{n,i} Q_C x_n\| : n \geq 0, i \geq 1 \Bigg\}. \end{aligned}$$

Then M' is a positive constant.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

In fact, using Lemma 2.1, we have, for $n \geq 0$,

$$\begin{aligned} &\left\| J_{r_{n+1}}^A\left(\frac{u_{n+1} + y_{n+1}}{2}\right) - J_{r_n}^A\left(\frac{u_n + y_n}{2}\right)\right\| \\ &\quad \leq \frac{\|u_{n+1} - u_n\|}{2} + \frac{\|y_{n+1} - y_n\|}{2} + 2\frac{|r_n - r_{n+1}|}{\varepsilon} M'. \end{aligned} \tag{2.7}$$

In view of (2.7) and Lemma 2.1, we have, for $n \geq 0$,

$$\begin{aligned} &\|u_{n+1} - u_n\| \\ &\quad \leq (1 - \delta_{n+1})\|y_{n+1} - y_n\| + |\delta_{n+1} - \delta_n|\|y_n\| \\ &\qquad + \delta_{n+1}\left\| J_{r_{n+1}}^A\left(\frac{u_{n+1} + y_{n+1}}{2}\right) - J_{r_n}^A\left(\frac{u_n + y_n}{2}\right)\right\| + |\delta_{n+1} - \delta_n|\left\| J_{r_n}^A\left(\frac{u_n + y_n}{2}\right)\right\| \\ &\quad \leq (1 - \delta_{n+1})\|y_{n+1} - y_n\| + 2|\delta_{n+1} - \delta_n| M' \\ &\qquad + \frac{\delta_{n+1}}{2}\left(\|u_{n+1} - u_n\| + \|y_{n+1} - y_n\|\right) + \frac{2}{\varepsilon}\delta_{n+1}|r_n - r_{n+1}| M', \end{aligned}$$

which implies that

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|y_{n+1}-y_n\| + \frac{4|\delta_{n+1}-\delta_n|M'}{2-\delta_{n+1}} + \frac{4\delta_{n+1}|r_n-r_{n+1}|M'}{(2-\delta_{n+1})\varepsilon} \\ &\le \|y_{n+1}-y_n\| + 4|\delta_{n+1}-\delta_n|M' + \frac{4M'}{\varepsilon}|r_{n+1}-r_n|. \end{aligned} \tag{2.8}$$

Now, in view of Lemma 2.1, computing the following:

$$\begin{aligned}&\|W_{n+1,i}Q_Cx_{n+1}-W_{n,i}Q_Cx_n\| \\ &\quad= \big\|\big[(1-\alpha_{n+1,i})J^A_{r_{n+1}}+\alpha_{n+1,i}S_i\big]Q_Cx_{n+1}-\big[(1-\alpha_{n,i})J^A_{r_n}+\alpha_{n,i}S_i\big]Q_Cx_n\big\| \\ &\quad\le (1-\alpha_{n+1,i})\big\|J^A_{r_{n+1}}Q_Cx_{n+1}-J^A_{r_n}Q_Cx_n\big\| \\ &\qquad+\alpha_{n+1,i}\|S_iQ_Cx_{n+1}-S_iQ_Cx_n\| \\ &\qquad+|\alpha_{n+1,i}-\alpha_{n,i}|\big(\|S_iQ_Cx_n\|+\big\|J^A_{r_n}Q_Cx_n\big\|\big) \\ &\quad\le (1-\alpha_{n+1,i})\|x_{n+1}-x_n\|+2M'\frac{|r_{n+1}-r_n|}{\varepsilon} \\ &\qquad+\alpha_{n+1,i}\|x_{n+1}-x_n\|+2M'|\alpha_{n+1,i}-\alpha_{n,i}| \\ &\quad= \|x_{n+1}-x_n\|+2M'\frac{|r_{n+1}-r_n|}{\varepsilon}+2M'|\alpha_{n+1,i}-\alpha_{n,i}|. \end{aligned} \tag{2.9}$$

Using (2.9), we have, for $n\ge 0$,

$$\begin{aligned}&\left\|\sum_{i=1}^{\infty}b_{n+1,i}W_{n+1,i}Q_Cx_{n+1}-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cx_n\right\| \\ &\quad\le \left\|\sum_{i=1}^{\infty}b_{n+1,i}W_{n+1,i}Q_Cx_{n+1}-\sum_{i=1}^{\infty}b_{n+1,i}W_{n,i}Q_Cx_n\right\| \\ &\qquad+\left\|\sum_{i=1}^{\infty}b_{n+1,i}W_{n,i}Q_Cx_n-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cx_n\right\| \\ &\quad\le \sum_{i=1}^{\infty}b_{n+1,i}\|W_{n+1,i}Q_Cx_{n+1}-W_{n,i}Q_Cx_n\| \\ &\qquad+\sum_{i=1}^{\infty}|b_{n+1,i}-b_{n,i}|\|W_{n,i}Q_Cx_n\| \\ &\quad\le \|x_{n+1}-x_n\|+2M'\frac{|r_{n+1}-r_n|}{\varepsilon}+2M'\sum_{i=1}^{\infty}|\alpha_{n+1,i}-\alpha_{n,i}| \\ &\qquad+M'\sum_{i=1}^{\infty}|b_{n+1,i}-b_{n,i}|. \end{aligned} \tag{2.10}$$

Using (2.10), we know that

$$\begin{aligned}&\|y_{n+1}-y_n\| \\ &\quad\le \beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\end{aligned}$$

$$
\begin{aligned}
&+ (1-\beta_{n+1})\left\| \sum_{i=1}^{\infty} b_{n+1,i} W_{n+1,i} Q_C x_{n+1} - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C x_n \right\| \\
&+ |\beta_{n+1}-\beta_n| \sum_{i=1}^{\infty} b_{n,i} \| W_{n,i} Q_C x_n \| \\
\le{}& \|x_{n+1}-x_n\| + 2|\beta_{n+1}-\beta_n| M' + 2M' \frac{|r_{n+1}-r_n|}{\varepsilon} \\
&+ 2M' \sum_{i=1}^{\infty} |\alpha_{n+1,i}-\alpha_{n,i}| + M' \sum_{i=1}^{\infty} |b_{n+1,i}-b_{n,i}|. \qquad (2.11)
\end{aligned}
$$

Thus in view of (2.8) and (2.11), we have, for $n \ge 0$,

$$
\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\le \gamma_{n+1}\eta k \|x_{n+1}-x_n\| + (1+\eta)|\gamma_{n+1}-\gamma_n| M' \\
&\quad + (1-\gamma_{n+1}\overline{\gamma}) \Bigg[\|x_n - x_{n+1}\| + 2M'|\beta_{n+1}-\beta_n| + 2M' \frac{|r_{n+1}-r_n|}{\varepsilon} \\
&\quad + M' \sum_{i=1}^{\infty} |b_{n+1,i}-b_{n,i}| + 2M' \sum_{i=1}^{\infty} |\alpha_{n+1,i}-\alpha_{n,i}| + 4M'|\delta_{n+1}-\delta_n| + \frac{4M'}{\varepsilon}|r_{n+1}-r_n| \Bigg] \\
&= \big[1-\gamma_{n+1}(\overline{\gamma}-\eta k)\big] \|x_{n+1}-x_n\| + (1+\eta)M'|\gamma_{n+1}-\gamma_n| \\
&\quad + (1-\gamma_{n+1}\overline{\gamma}) \Bigg[2M' \sum_{i=1}^{\infty} |\alpha_{n+1,i}-\alpha_{n,i}| + 2M'|\beta_{n+1}-\beta_n| \\
&\quad + 4M'|\delta_{n+1}-\delta_n| + M' \sum_{i=1}^{\infty} |b_{n+1,i}-b_{n,i}| + \frac{6M'}{\varepsilon}|r_{n+1}-r_n| \Bigg]. \qquad (2.12)
\end{aligned}
$$

Using Lemma 1.4, we have from (2.12) $\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0$. Since $\gamma_n \to 0$, we have $x_{n+1} - u_n = \gamma_n(\eta f(x_n) - Tu_n) \to 0$, as $n \to \infty$, which implies that $x_n - u_n \to 0$, as $n \to \infty$.

Since $\delta_n \to 0$, we have $u_n - y_n = \delta_n [J_{r_n}^A(\frac{u_n+y_n}{2}) - y_n] \to 0$, as $n \to \infty$. Therefore, $x_n - y_n \to 0$, as $n \to \infty$.

Then

$$
\begin{aligned}
&\left\| y_n - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right\| \\
&\le \left\| y_n - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C x_n \right\| + \left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} (Q_C x_n - Q_C y_n) \right\| \\
&\le \beta_n \left\| x_n - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C x_n \right\| + \|x_n - y_n\| \to 0,
\end{aligned}
$$

since $\beta_n \to 0$. Moreover, $x_{n+1} - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \to 0$, as $n \to \infty$.

Step 3. $\limsup_{n\to+\infty} \langle \eta f(q_0) - Tq_0, J(x_{n+1}-q_0)\rangle \le 0$.

From Lemma 2.5, we know that $v_t = t\eta f(v_t) + (I - tT)\sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t$ for $t \in (0,1)$. Moreover, $v_t \to q_0 \in \Omega$, as $t \to 0$. q_0 is the unique solution of the variational inequality (2.3).

Since $\|v_t\| \le \|v_t - q_0\| + \|q_0\|$, we see that $\{v_t\}$ is bounded, as $t \to 0$. Using Lemma 1.3, we have

$$\begin{aligned}
&\|v_t - y_n\|^2 \\
&\quad = \left\| v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n + \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n - y_n \right\|^2 \\
&\quad \le \left\| v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right\|^2 + 2\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n - y_n, J(v_t - y_n) \right\rangle \\
&\quad = \left\| t\eta f(v_t) + (I - tT) \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right\|^2 \\
&\qquad + 2\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n - y_n, J(v_t - y_n) \right\rangle \\
&\quad \le \left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right\|^2 \\
&\qquad + 2t\left\langle \eta f(v_t) - T\sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t, J\left(v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right) \right\rangle \\
&\qquad + 2\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n - y_n, J(v_t - y_n) \right\rangle \\
&\quad \le \|v_t - y_n\|^2 + 2t\left\langle \eta f(v_t) - T\sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t, J\left(v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right) \right\rangle \\
&\qquad + 2\left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n - y_n \right\| \|v_t - y_n\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&t\left\langle T\sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t - \eta f(v_t), J\left(v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right) \right\rangle \\
&\quad \le \left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n - y_n \right\| \|v_t - y_n\|.
\end{aligned}$$

So, $\lim_{t\to 0} \limsup_{n\to+\infty} \langle T\sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t - \eta f(v_t), J(v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n)\rangle \le 0$ in view of Step 2.

Since $v_t \to q_0$, we have $\sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C v_t \to \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C q_0 = Q_C q_0 = q_0$, as $t \to 0$. Noticing the fact that

$$\begin{aligned}
&\left\langle Tq_0 - \eta f(q_0), J\left(q_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right) \right\rangle \\
&\quad = \left\langle Tq_0 - \eta f(q_0), J\left(q_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right) - J\left(v_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} Q_C y_n \right) \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\left\langle Tq_0-\eta f(q_0),J\left(v_t-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n\right)\right\rangle\\
&=\left\langle Tq_0-\eta f(q_0),J\left(q_0-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n\right)-J\left(v_t-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n\right)\right\rangle\\
&\quad+\left\langle Tq_0-\eta f(q_0)-T\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cv_t+\eta f(v_t),J\left(v_t-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n\right)\right\rangle\\
&\quad+\left\langle T\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cv_t-\eta f(v_t),J\left(v_t-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n\right)\right\rangle,
\end{aligned}
$$

we have $\limsup_{n\to+\infty}\langle Tq_0-\eta f(q_0),J(q_0-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n)\rangle\le 0$.

Since $\langle Tq_0-\eta f(q_0),J(q_0-x_{n+1})\rangle=\langle Tq_0-\eta f(q_0),J(q_0-x_{n+1})-J(q_0-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n)\rangle+\langle Tq_0-\eta f(q_0),J(q_0-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n)\rangle$ and $x_{n+1}-\sum_{i=1}^{\infty}b_{n,i}W_{n,i}Q_Cy_n\to 0$ in view of Step 2, we have $\limsup_{n\to\infty}\langle \eta f(q_0)-Tq_0,J(x_{n+1}-q_0)\rangle\le 0$.

Step 4. $x_n\to q_0$, as $n\to+\infty$, where $q_0\in\Omega$ is the same as that in Step 3.

Since

$$
\|u_n-q_0\|\le(1-\delta_n)\|y_n-q_0\|+\delta_n\left\|\frac{y_n+u_n}{2}-q_0\right\|,
$$

we have

$$
\|u_n-q_0\|\le\|y_n-q_0\|\le\beta_n\|x_n-q_0\|+(1-\beta_n)\|x_n-q_0\|=\|x_n-q_0\|.
$$

Using Lemma 1.3, we have, for $n\ge 0$,

$$
\begin{aligned}
&\|x_{n+1}-q_0\|^2\\
&\quad=\bigl\|\gamma_n\bigl(\eta f(x_n)-Tq_0\bigr)+(I-\gamma_nT)(u_n-q_0)\bigr\|^2\\
&\quad\le(1-\gamma_n\overline{\gamma})^2\|u_n-q_0\|^2+2\gamma_n\bigl\langle\eta f(x_n)-Tq_0,J(x_{n+1}-q_0)\bigr\rangle\\
&\quad\le(1-\gamma_n\overline{\gamma})^2\|x_n-q_0\|^2\\
&\qquad+2\gamma_n\eta\bigl\langle f(x_n)-f(q_0),J(x_{n+1}-q_0)-J(x_n-q_0)\bigr\rangle\\
&\qquad+2\gamma_n\eta\bigl\langle f(x_n)-f(q_0),J(x_n-q_0)\bigr\rangle+2\gamma_n\bigl\langle\eta f(q_0)-Tq_0,J(x_{n+1}-q_0)\bigr\rangle\\
&\quad\le\bigl[1-\gamma_n(\overline{\gamma}-2\eta k)\bigr]\|x_n-q_0\|^2\\
&\qquad+2\gamma_n\bigl[\bigl\langle\eta f(q_0)-Tq_0,J(x_{n+1}-q_0)\bigr\rangle+\eta\|x_n-q_0\|\|x_{n+1}-x_n\|\bigr].
\end{aligned}
\tag{2.13}
$$

Let $\delta_n^{(1)}=\gamma_n(\overline{\gamma}-2\eta k)$, $\delta_n^{(2)}=2\gamma_n[\langle\eta f(q_0)-Tq_0,J(x_{n+1}-q_0)\rangle+\eta\|x_n-q_0\|\|x_{n+1}-x_n\|]$. Then (2.13) can be simplified as $\|x_{n+1}-p_0\|^2\le(1-\delta_n^{(1)})\|x_n-q_0\|^2+\delta_n^{(2)}$.

Using the assumption (ii), the result of Steps 2 and 3 and Lemma 1.4, we know that $x_n\to q_0$, as $n\to+\infty$.

Combining with the result of Step 2, $y_n\to q_0$ and $u_n\to q_0$, as $n\to\infty$.

Step 5. $z_n\to q_0$, as $n\to\infty$.

Since $\sum_{k=1}^{n}a_k\to\infty$ and $\|x_n-q_0\|\to 0$, as $n\to\infty$, we have, for $\forall\varepsilon>0$, there exists N^*, such that, for all $n\ge N^*$, $\frac{1}{\sum_{k=1}^{n}a_k}\sum_{k=1}^{N^*}a_kM''<\frac{\varepsilon}{2}$ and $\|x_n-q_0\|<\frac{\varepsilon}{2}$, where $M''=\max\{\|x_k-$

$q_0\| : k = 1, 2, \dots, N^*\}$. Then, for all $n > N^*$,

$$\begin{aligned}
\|z_n - q_0\| &= \left\| \frac{1}{\sum_{k=1}^n a_k} \sum_{k=1}^n a_k(x_k - q_0) \right\| \\
&= \left\| \frac{1}{\sum_{k=1}^n a_k} \left[\sum_{k=1}^{N^*} a_k(x_k - q_0) + \sum_{k=N^*+1}^{n} a_k(x_k - q_0) \right] \right\| \\
&\leq \frac{1}{\sum_{k=1}^n a_k} \sum_{k=1}^{N^*} a_k\|x_k - q_0\| + \frac{1}{\sum_{k=1}^n a_k} \sum_{k=N^*+1}^{n} a_k\|x_k - q_0\| \\
&\leq \frac{1}{\sum_{k=1}^n a_k} \sum_{k=1}^{N^*} a_k M'' + \frac{\sum_{k=N^*+1}^n a_k}{\sum_{k=1}^n a_k} \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which implies that $z_n \to q_0$, as $n \to \infty$.

This completes the proof. □

Remark 2.6 The assumptions imposed on the real number sequences in Theorem 2.2 are reasonable if we take $\alpha_{n,i} = \frac{1}{(n+1)i^2}$, $\gamma_n = \delta_n = \beta_n = \frac{1}{1+n}$, $r_n = \varepsilon + \frac{1}{n+1}$, and $b_{n,i} = \frac{n+1}{(n+2)^i}$, for $n \geq 0$ and $i = 1, 2, \dots$.

2.2 Ergodic convergence of the second iterative algorithm

Theorem 2.7 *Let E, Ω, f, A, S_i, $\{r_n\}$, $\{\delta_n\}$, $\{b_{n,i}\}$, and $\{\alpha_{n,i}\}$ be the same as those in Theorem 2.2. Let C be a nonempty, closed, and convex subset of E. Suppose $\Omega \neq \emptyset$, $0 < k < \frac{1}{2}$, the duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero and $\{\zeta_n\} \subset (0,1)$. Let $\{x_n\}$ be generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C, \\ u_n = (1 - \delta_n)x_n + \delta_n J_{r_n}^A(\frac{u_n+x_n}{2}), \\ x_{n+1} = [\zeta_n f + (1 - \zeta_n)I] \sum_{i=1}^{\infty} b_{n,i}[(1 - \alpha_{n,i})J_{r_n}^A + \alpha_{n,i}S_i]u_n, \quad n \geq 0. \end{cases} \tag{B}$$

Further suppose that the following conditions are satisfied:

(vi) $\sum_{n=0}^{\infty} \zeta_n = \infty$, $\sum_{n=0}^{\infty} |\zeta_{n+1} - \zeta_n| < +\infty$, *and* $\zeta_n \to 0$, *as* $n \to \infty$.

Then both $\{u_n\}$ and $\{x_n\}$ converge strongly to the unique element $p_0 \in \Omega$, which satisfies the following variational inequality: *for* $\forall y \in \Omega$,

$$\langle p_0 - f(p_0), J(p_0 - y) \rangle \leq 0. \tag{2.14}$$

Moreover, the ergodic convergence is obtained in the sense that

$$z_n := \frac{1}{\sum_{k=1}^n a_k} \sum_{k=1}^n a_k x_k, \quad n \geq 1,$$

converges strongly to the above p_0 under the assumption that $\{a_n\}$ is a sequence of positive numbers such that $\sum_{k=1}^n a_k \to \infty$, as $n \to \infty$.

Proof Using the same method as that in Theorem 2.2, we know that $\{u_n\}$ is well-defined. Let $V_n = \zeta_n f + (1-\zeta_n)I$.

Step 1. $V_n : C \to C$ is a contraction.

For $\forall x, y \in C$,

$$\begin{aligned}\|V_n x - V_n y\| &= \big\|\zeta_n\big(f(x) - f(y)\big) + (1-\zeta_n)(x-y)\big\| \\ &\leq \zeta_n k\|x-y\| + (1-\zeta_n)\|x-y\| = \big[1-(1-k)\zeta_n\big]\|x-y\|. \end{aligned} \tag{2.15}$$

Step 2. $\{x_n\}$ is bounded.

Let $W_{n,i}$ be the same as that in Lemma 2.3 and Theorem 2.2, then for $p \in \Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_{n,i})$, we can easily know that $\|u_n - p\| \leq \|x_n - p\|$. Using (2.15), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \left\| V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} p + V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} p - p \right\| \\ &\leq \big[1-(1-k)\zeta_n\big]\|u_n - p\| + \zeta_n \big\|f(p) - p\big\| \\ &\leq \big[1-(1-k)\zeta_n\big]\|x_n - p\| + \zeta_n \big\|f(p) - p\big\| \\ &\leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-k}\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Thus $\{J_{r_n}^A(\frac{u_n + x_n}{2})\}$, $\{u_n\}$, $\{W_{n,i}u_n\}$ and $\{f(\sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n)\}$ are all bounded for $n \geq 0$ and $i = 1, 2, \ldots$. Let

$$\begin{aligned} M''' = \sup\Bigg\{ & M', \|x_n\|, \|u_n\|, \left\| J_{r_n}^A\left(\frac{u_n + x_n}{2}\right)\right\|, \|W_{n,i}u_n\|, \\ & \left\| f\left(\sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n\right)\right\| : n \geq 0, i = 1, 2, \ldots \Bigg\},\end{aligned}$$

where M' is the same as that in Theorem 2.2, then M''' is a positive constant.

Step 3. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

In fact, using Lemma 2.1, we know that

$$\begin{aligned} &\|u_{n+1} - u_n\| \\ &\quad\leq (1-\delta_{n+1})\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|x_n\| + \delta_{n+1}\left\| J_{r_{n+1}}^A\left(\frac{u_{n+1} + x_{n+1}}{2}\right) - J_{r_n}^A\left(\frac{u_n + x_n}{2}\right)\right\| \\ &\qquad + |\delta_{n+1} - \delta_n|\left\| J_{r_n}^A\left(\frac{u_n + x_n}{2}\right)\right\| \\ &\quad\leq (1-\delta_{n+1})\|x_{n+1} - x_n\| + 2M'''|\delta_{n+1} - \delta_n| + \delta_{n+1}\left(\frac{\|u_{n+1} - u_n\|}{2} + \frac{\|x_{n+1} - x_n\|}{2}\right) \\ &\qquad + \frac{2M'''}{\varepsilon}|r_{n+1} - r_n|,\end{aligned}$$

which implies that $\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + 4|\delta_{n+1} - \delta_n|M''' + 4M''' \frac{|r_{n+1} - r_n|}{\varepsilon}$.

Thus using (2.15) and noticing the result of (2.9), we have, for $n \geq 0$,

$$
\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\quad = \left\| V_{n+1} \sum_{i=1}^{\infty} b_{n+1,i} W_{n+1,i} u_{n+1} - V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\| \\
&\quad \leq \left\| V_{n+1} \sum_{i=1}^{\infty} b_{n+1,i} W_{n+1,i} u_{n+1} - V_{n+1} \sum_{i=1}^{\infty} b_{n+1,i} W_{n,i} u_n \right\| \\
&\qquad + \left\| V_{n+1} \sum_{i=1}^{\infty} b_{n+1,i} W_{n,i} u_n - V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\| \\
&\quad \leq \bigl[1 - (1-k)\zeta_{n+1}\bigr] \sum_{i=1}^{\infty} b_{n+1,i} \| W_{n+1,i} u_{n+1} - W_{n,i} u_n \| \\
&\qquad + \left\| V_{n+1} \sum_{i=1}^{\infty} b_{n+1,i} W_{n,i} u_n - V_{n+1} \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\| + \left\| (V_{n+1} - V_n) \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\| \\
&\quad \leq \bigl[1 - (1-k)\zeta_{n+1}\bigr] \sum_{i=1}^{\infty} b_{n+1,i} \left[\|u_{n+1} - u_n\| + \frac{2M'''}{\varepsilon} |r_{n+1} - r_n| + 2M''' |\alpha_{n+1,i} - \alpha_{n,i}| \right] \\
&\qquad + \left\| \zeta_{n+1} f\left(\sum_{i=1}^{\infty} b_{n+1,i} W_{n,i} u_n \right) + (1 - \zeta_{n+1}) \sum_{i=1}^{\infty} b_{n+1,i} W_{n,i} u_n \right. \\
&\qquad \left. - \zeta_{n+1} f\left(\sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) - (1 - \zeta_{n+1}) \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\| \\
&\qquad + |\zeta_{n+1} - \zeta_n| \left\| f\left(\sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\| + |\zeta_{n+1} - \zeta_n| \left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\| \\
&\quad \leq \bigl[1 - (1-k)\zeta_{n+1}\bigr] \|x_{n+1} - x_n\| + 2M''' \sum_{i=1}^{\infty} |\alpha_{n+1,i} - \alpha_{n,i}| + \frac{6M'''}{\varepsilon} |r_{n+1} - r_n| \\
&\qquad + 4M''' |\delta_{n+1} - \delta_n| + 2M''' |\zeta_{n+1} - \zeta_n| + M''' \sum_{i=1}^{\infty} |b_{n+1,i} - b_{n,i}|.
\end{aligned}
$$

Then Lemma 1.4 implies that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $\zeta_n \to 0$, we have $x_{n+1} - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \to 0$, as $n \to \infty$. Since $\delta_n \to 0$, we have $x_n - u_n \to 0$, which implies that $\sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n \to 0$, as $n \to \infty$.

Step 4. $\limsup_{n\to+\infty} \langle f(p_0) - p_0, J(x_{n+1} - p_0)\rangle \leq 0$.

Noticing Lemmas 1.7 and 2.3, we know that there exists z_t such that $z_t = tf(z_t) + (1 - t)\sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t$ for $t \in (0,1)$. Moreover, $z_t \to p_0 \in \Omega$, as $t \to 0$. p_0 is the unique solution of the variational inequality (2.14).

Since $\|z_t\| \leq \|z_t - p_0\| + \|p_0\|$, $\{z_t\}$ is bounded, as $t \to 0$. Using Lemma 1.3, we have

$$
\begin{aligned}
&\|z_t - u_n\|^2 \\
&\quad = \left\| z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n + \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\| z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\|^2 + 2\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n, J(z_t - u_n) \right\rangle \\
&= \left\| tf(z_t) + (1-t) \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right\|^2 \\
&\quad + 2\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n, J(z_t - u_n) \right\rangle \\
&\leq \|z_t - u_n\|^2 + 2t\left\langle f(z_t) - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t, J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\quad + 2\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n, J(z_t - u_n) \right\rangle \\
&\leq \|z_t - u_n\|^2 + 2t\left\langle f(z_t) - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t, J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\quad + 2\left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n \right\| \|z_t - u_n\|,
\end{aligned}
$$

which implies that

$$
t\left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t - f(z_t), J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \leq \left\| \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - u_n \right\| \|z_t - u_n\|.
$$

So, $\lim_{t\to 0} \limsup_{n\to+\infty} \langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t - f(z_t), J(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n) \rangle \leq 0$ in view of Step 3.

Since $z_t \to p_0$, we have $\sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t \to \sum_{i=1}^{\infty} b_{n,i} W_{n,i} p_0 = p_0$, as $t \to 0$. Noticing the fact that

$$
\begin{aligned}
&\left\langle p_0 - f(p_0), J\left(p_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\quad = \left\langle p_0 - f(p_0), J\left(p_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) - J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\qquad + \left\langle p_0 - f(p_0), J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\quad = \left\langle p_0 - f(p_0), J\left(p_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) - J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\qquad + \left\langle p_0 - f(p_0) - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t + f(z_t), J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle \\
&\qquad + \left\langle \sum_{i=1}^{\infty} b_{n,i} W_{n,i} z_t - f(z_t), J\left(z_t - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \right) \right\rangle,
\end{aligned}
$$

we have $\limsup_{n\to+\infty} \langle p_0 - f(p_0), J(p_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n) \rangle \leq 0$.

Since $\langle p_0 - f(p_0), J(p_0 - x_{n+1})\rangle = \langle p_0 - f(p_0), J(p_0 - x_{n+1}) - J(p_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n)\rangle + \langle p_0 - f(p_0), J(p_0 - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n)\rangle$ and $x_{n+1} - \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n \to 0$ in view of Step 3, we have $\limsup_{n\to\infty} \langle f(p_0) - p_0, J(x_{n+1} - p_0)\rangle \le 0$.

Step 5. $x_n \to p_0$, as $n \to +\infty$, where $p_0 \in \Omega$ is the same as in Step 4.

Since

$$\|u_n - p_0\| \le (1 - \delta_n)\|x_n - p_0\| + \delta_n \left\|\frac{x_n + u_n}{2} - p_0\right\|,$$

we have

$$\|u_n - p_0\| \le \|x_n - p_0\|.$$

Using Lemma 1.3, we have, for $n \ge 0$,

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad = \left\| V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - p_0 \right\|^2 \\
&\quad = \left\| V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} u_n - V_n \sum_{i=1}^{\infty} b_{n,i} W_{n,i} p_0 + V_n p_0 - p_0 \right\|^2 \\
&\quad \le \bigl[1 - (1-k)\zeta_n\bigr]\|u_n - p_0\|^2 + 2\bigl\langle V_n p_0 - p_0, J(x_{n+1} - p_0)\bigr\rangle \\
&\quad \le \bigl[1 - (1-k)\zeta_n\bigr]\|x_n - p_0\|^2 + 2\zeta_n\bigl\langle f(p_0) - p_0, J(x_{n+1} - p_0)\bigr\rangle.
\end{aligned}$$

Using Lemma 1.4, the assumptions and the result of Step 4, we know that $x_n \to p_0$, as $n \to +\infty$. Combining with the result of Step 3, $u_n \to p_0$, as $n \to \infty$.

Copy Step 5 in Theorem 2.2, $z_n \to p_0$, as $n \to \infty$.

This completes the proof. □

Remark 2.8 The assumptions imposed on the real number sequences in Theorem 2.7 are reasonable if we take $\alpha_{n,i} = \frac{1}{(n+1)i^2}$, $\delta_n = \zeta_n = \frac{1}{n+1}$, $b_{n,i} = \frac{n+1}{(n+2)^i}$, and $r_n = \varepsilon + \frac{1}{n+1}$, for $n \ge 0$ and $i = 1, 2, \ldots$.

Remark 2.9 The four sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ in Theorem 2.2 and the three sequences $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ in Theorem 2.7 are proved to be strongly convergent to the zero point of an m-accretive mapping and the fixed point of an infinite family of non-expansive mappings. The strongly convergent point is proved to be the unique solution of one kind variational inequalities.

Remark 2.10 In Theorem 2.7, V_n can be regarded as an averaged mapping, whose definition can be seen in [16].

Remark 2.11 The discussions on Theorems 2.2 and 2.7 are undertaken in the frame of a real smooth and uniformly convex Banach space, which is more general than that in Hilbert space.

3 Examples and numerical experiments

In this section, we provide some numerical experiments to show that both algorithms (A) and (B) are effective. In our experiments, we consider the following examples.

Example 3.1 In algorithm (A), suppose $E = C = (-\infty, +\infty)$. Let $a_i = 1$, $\alpha_{n,i} = \frac{1}{2^{n+i}}$, $b_{n,i} = \frac{n+1}{(n+2)^i}$, $\gamma_n = \delta_n = \beta_n = \frac{1}{n+1}$, $r_n = \frac{n+2}{n+1}$, $\varepsilon = 1$, $f(x) = \frac{x}{14}$, $k = \frac{1}{7}$, $Ax = \frac{x}{2}$, $Tx = \frac{2x}{7}$, $\overline{\gamma} = \frac{2}{7}$, $\eta = \frac{1}{2}$, and $S_i x = \frac{x}{2^i}$. Then all of the assumptions in Theorem 2.2 are satisfied. And $\Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap A^{-1}0 = \{0\}$.

Table 1 The values of $\{y_n\}$, $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ with initial value $x_0 = -4$

n	y_n	u_n	x_n	z_n
0	−4.0000000000	−1.333333333333333	−4.0000000000000	
1	−0.8477427	−0.635807052254677	−1.0952380952381	−1.0952380952381
2	−0.408616	−0.348080327113469	−0.564535296490403	−0.829886695864249
3	−0.2269676	−0.203325144325693	−0.321650478060882	−0.660474623263127
4	−0.1335324	−0.122849839925766	−0.191673798999402	−0.543274417197196
5	−0.08100538	−0.075755030882579	−0.117198947637147	−0.458059323285186
6	−0.05007082	−0.047345876009489	−0.072845261243154	−0.393856979611514
7	−0.03134004	−0.029870972328354	−0.045785050770546	−0.344132418348518
8	−0.01979206	−0.018977571393789	−0.029008549436138	−0.304741934734471
9	−0.01258285	−0.012121482143799	−0.018490222101272	−0.272936188886338
10	−0.008041086	−0.007775264661297	−0.011841190590053	−0.246826689056709
11	−0.005159962	−0.005004685808672	−0.007611755159024	−0.225079876884192
12	−0.003322376	−0.003230633801933	−0.004908180655962	−0.20673223553184
13	−0.002145251	−0.002090525234888	−0.003173114874022	−0.191073841635085
14	−0.001388506	−0.001355593524252	−0.002055956135385	−0.177572564099392
15	−0.000900557	−0.000880623111319	−0.001334667828875	−0.165823371014691
16	−0.000585129	−0.000572980669566	−0.000867876867878	−0.155513652630515
17	−0.000380779	−0.000373335337785	−0.000565174013077	−0.146399036241254
18	−0.000248140	−0.000243557044559	−0.000368530757053	−0.13828623038102
19	−0.000619034	−0.000159070048612	−0.000240587259597	−0.131020670216735
20	−0.000105754	−0.000103995450936	−0.000157227239453	−0.124477498067871
21	−0.0000691467	−0.000068051443395	−0.000102847940106	−0.118554895680834
22	−0.0000452522	−0.000044567863803	−0.000067334619358	−0.113169097450767
23	−0.0000296392	−0.0000292103517604	−0.000044118782606	−0.108250620117369
24	−0.0000194276	−0.000019158216414	−0.000028928262428	−0.103741382956746
25	−0.0000127431	−0.000012573398811	−0.000018980591458	−0.099592486862135
26	−0.0000083638	−0.000008256737423	−0.000012461301835	−0.095762485879046
27	−0.0000054928	−0.000005425047229	−0.000008185847743	−0.092216030322332
28	−0.0000036092	−0.000003566325208	−0.000005380130737	−0.088922792815489
29	−0.0000023728	−0.000002345554771	−0.000003537814776	−0.085856611608568
30	−0.0000015607	−0.000001543347773	−0.000002327427839	−0.082994802135877
31	0.0000000000	0.000000000000000	−0.000001531804738	−0.080317599867130

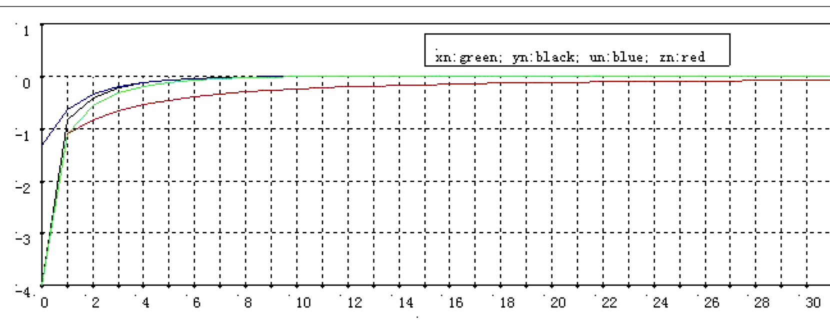


Figure 1 The convergence of $\{y_n\}$, $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$.

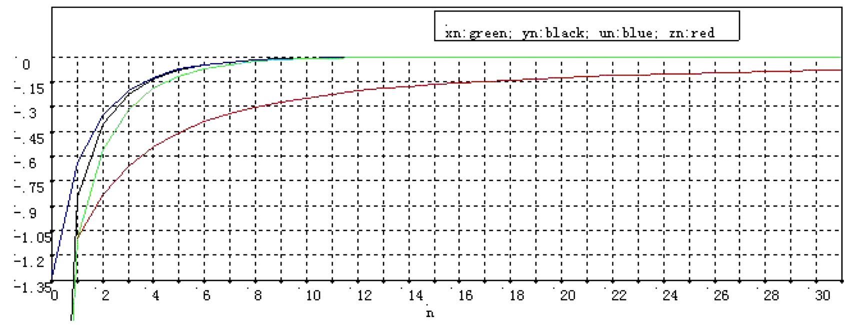


Figure 2 Screenshot of Figure 1.

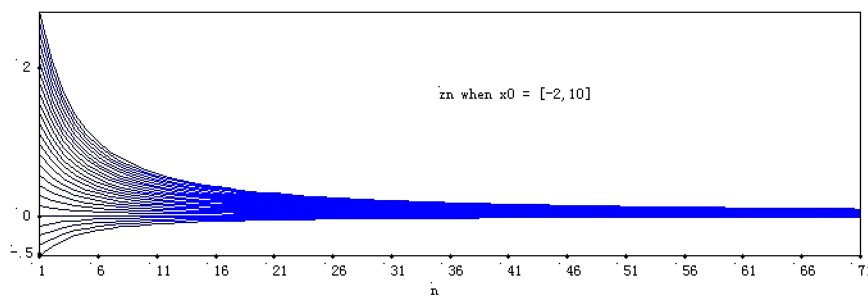


Figure 3 The convergence of $\{z_n\}$ under different initial values.

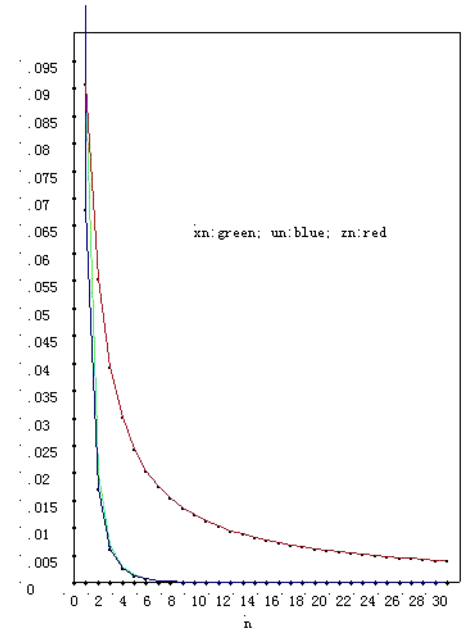
Table 2 The values of $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ with initial value $x_0 = 8$

n	x_n	u_n	z_n
0	8.0000000000000	2.6666666666667	
1	0.0907029478458	0.0680272108844	0.09070295
2	0.0199727386316	0.0170138143899	0.05533785
3	0.0068807569080	0.0061640113968	0.03918548
4	0.0028754047605	0.0026453723797	0.03010796
5	0.0013373257810	0.0012506472582	0.02435383
6	0.0006653502921	0.0006291407524	0.02040576
7	0.0003466462534	0.0003303972103	0.01754017
8	0.0001867447646	0.0001790597949	0.01537099
9	0.0001031871958	0.0000994036653	0.01367457
10	0.0000581631893	0.0000562404392	0.01231293
11	0.0000333151231	0.0000323125847	0.01119660
12	0.0000193365400	0.0000188025922	0.01026516
13	0.0000113483809	0.0000110588814	0.00947641
14	0.0000067234088	0.0000065640391	0.00879999
15	0.0000040158582	0.0000039269655	0.00821360
16	0.0000024157047	0.0000023655516	0.00770040
17	0.0000014622202	0.0000014336377	0.00724752
18	0.0000008899717	0.0000008735364	0.00684493
19	0.0000005443463	0.0000005348202	0.00648470
20	0.0000003344191	0.0000003288581	0.00616048

Remark 3.1 All codes were written in Visual Basic Six. For the initial value $x_0 = -4$, the values of $\{y_n\}$, $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ with different n are reported in Table 1.

Remark 3.2 In Figure 1, the abscissa denotes the iterative step and the ordinate denotes the values of $\{y_n\}$, $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ with different iterative step n.

Figure 4 The convergence of $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$.



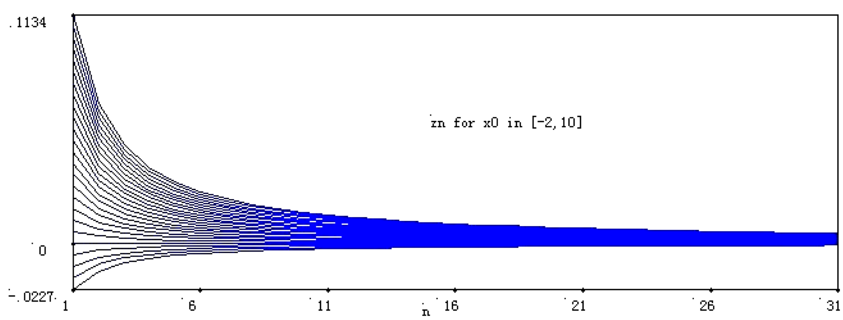


Figure 5 The convergence of $\{z_n\}$ for different initial values.

Remark 3.3 Table 1 and Figure 1 show that the sequences $\{y_n\}$, $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ converge to 0. Also, $\{0\} = \Omega$.

Remark 3.4 Figure 2 is a screen shot of Figure 1, whose ordinates are enlarged. Our purpose to draw Figure 2 is to let Figure 1 be clearer.

Remark 3.5 Figure 3 shows the values of ergodic sequence $\{z_n\}$ with the initial value x_0 being chosen arbitrarily in $[-2, 10]$.

Example 3.2 In algorithm (B), suppose $E = (-\infty, +\infty)$ and $C = [-2, 10]$. Let $a_i = 1$, $\alpha_{n,i} = \frac{1}{2^{n+i}}$, $b_{n,i} = \frac{n+1}{(n+2)^i}$, $\delta_n = \zeta_n = \frac{1}{n+1}$, $r_n = \frac{n+2}{n+1}$, $\varepsilon = 1$, $f(x) = \frac{x}{14}$, $k = \frac{1}{7}$, $Ax = \frac{x}{2}$, and $S_i x = \frac{x}{2^i}$. Then all of the assumptions in Theorem 2.7 are satisfied. Also $\Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap A^{-1}0 = \{0\}$.

All codes were written in Visual Basic Six, the values of $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ with different n are reported in Table 2.

Remark 3.6 Table 2 and Figure 4 show that the sequences $\{u_n\}$, $\{x_n\}$, and $\{z_n\}$ converge to 0. Also, $\{0\} = \Omega$.

Remark 3.7 Figure 5 shows the values of ergodic sequence $\{z_n\}$ with the initial value x_0 being chosen arbitrarily in $[-2, 10]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first and the third authors are responsible for the first and the second sections. The second author is responsible for the experiment in Section 3. All authors read and approve the final manuscript.

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